

NEW OSCILLATION CRITERIA FOR GENERAL NEUTRAL DELAY THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The aim of this paper is to study the oscillatory behavior of solutions of general third order neutral delay differential equations using a generalized Riccati transformation. We establish new sufficient conditions for oscillations of solutions. Illustrative examples are given to support our main results. The obtained results extend and improve some known criteria in the literature.

KEYWORDS: Third Order, Neutral Differential Equation, Oscillation Criteria

1. INTRODUCTION

There has been considerable interest in studying the oscillation of solutions of neutral delay differential equations in the last two decades. Although the oscillation of third-order equations has received less attentions relatively comparing with those of second-order, however there is an increasing interest in studying the oscillation of neutral delay third-order equations (see [2], [4], [6], [8])

The aim of this paper is to study the oscillation of solutions of the third order neutral differential equation

$$(a(t)[b(t)z'(t)]')' + f(t, z(t), z'(t)) = 0 \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$

under the assumptions

$$(H_1) \quad a(t), b(t), p(t) \in C((0, \infty)), \quad 0 \leq p(t) \leq p_0 < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} a^{-1}(s) ds = \infty, \quad \int_{t_0}^{\infty} b^{-1}(s) ds = \infty.$$

$$(H_2) \quad \forall t \geq t_0, \quad \sigma(t), \tau(t) \in C^1([t_0, \infty), R), \quad \sigma'(t) > 0, \quad \sigma(t) \leq t, \quad \tau(t) \leq t,$$

$$\lim_{t \rightarrow \infty} \sigma(t) = \infty, \quad \lim_{t \rightarrow \infty} \tau(t) = \infty, \quad \tau'(t) = \tau_0 > 0, \quad \sigma \circ \tau = \tau \circ \sigma$$

$$(H_3) \quad f \in C(R \times R^2, R), \quad \text{and} \quad \frac{f(t, u, v)}{v} \geq K > 0.$$

Many efforts were done to give sufficient conditions for oscillation of differential equations of the type (1.1) see ([6], [8], [10], [11]). To the best of our knowledge most of those papers consider a common condition on the nonlinear function f , namely

$$\frac{f(x)}{x} \geq k > 0 \tag{c}$$

One of our main goal of this paper is to establish new oscillation criteria for Eq.(1.1) without the traditional condition (c). The paper is organized as follows. In sec. 2 we give our main results, we establish sufficient conditions guarantee the oscillation of Eq.(1.1). In sec.3 we give some examples for which our criteria apply while the others in the literature fail.

As in [9], we say that a function $\phi(t, s, l)$ belongs to the function class Ω , denoted by $\phi \in \Omega$ if $\phi \in C(E, R)$, where $E = \{(t, s, l) : t_0 \leq l \leq s \leq t < \infty\}$, which satisfies $\phi(t, t, l) > 0$, $\phi(t, l, l) = 0$, and $\phi(t, s, l) > 0$ for $l < s < t$, and has the partial derivative $\frac{\partial \phi}{\partial s}$, defined by

$$\frac{\partial \phi}{\partial s} = \varphi(t, s, l) \phi(t, s, l), \quad \varphi \in \Omega \tag{1.5}$$

Further we define the operator $A[.; l, t]$ by

$$A[g; l, t] := \int_l^t \phi^2(t, s, l) g(s) ds \text{ for } t \geq s \geq l \geq t_0, \text{ where } g(s) \in C[t_0, \infty), \tag{1.6}$$

It is easy to see that $A[.; l, t]$ is a linear operator and satisfies

$$A[g'; l, t] = -2A[g\varphi; l, t] \text{ for } g \in C^1[t_0, \infty).$$

In what follows we use the notation of Philos-Kameneve type

$D = \{(t, s) : t_0 \leq s < t < \infty\}$ and $D_0 = \{(t, s) : t_0 \leq s \leq t < \infty\}$. We say that a continuous function $H : D \rightarrow [0, \infty)$ belongs to the function class X denoted by $H \in X$ if

- $H(t, t) = 0, H(t, s) > 0$ for $(t, s) \in D_0$.
- $H(t, s)$ has a continuous partial derivative with respect to s defined by

$$\frac{\partial H(t, s)}{\partial s} = -h(t, s) \sqrt{H(t, s)} \text{ for some } h \in C(D_0, R). \tag{1.7}$$

The key idea of the proofs depends of the technique used in [7, 9].

2. MAIN RESULTS

Before starting our main results, we begin with the following lemma which plays an important role in the proof of main results.

Lemma 2.1

If $x(t)$ is a nonoscillatory solution of Eq. (1.1), then either $z(t) > 0, z'(t) > 0, a(t)[b(t)z'(t)]' > 0$, eventually or $z(t) < 0, z'(t) < 0, a(t)[b(t)z'(t)]' < 0$, eventually where

$z(t)$ be defined as in (1.1).

Proof

Without loss of generality we assume that $x(t) > 0, x(\tau(t)) > 0$ for $t \geq t_0$. The proof when $x(t) < 0$ is similar. Now since $x(t) > 0, x(\tau(t)) > 0$, Then $z(t) > 0$. Firstly we claim that $a(t)[b(t)z'(t)]'$ is monotone and of one signe. If this is false, then there exists $t_1 \geq t_0$ such that $a(t)[b(t)z'(t)]' = 0$, for some $t \geq t_1$.

Since $a(t) > 0$, then $b(t)z'(t) = c$, c is constant

If $c = 0$, we have $z'(t) = 0 = z''(t)$ for some $t \geq t_1$,

which leads to $z(t) = 0$, this contradicts $z(t) > 0$.

Now for $c \neq 0$, we have $a(t)[b(t)z'(t)]' = -f(t, z(t), z'(t)) = 0$.

this contradicts the assumption (H_3) . Thus $a(t)[b(t)z'(t)]'$ is monotone and of one signe. Hence either $a(t)[b(t)z'(t)]' > 0$, or $a(t)[b(t)z'(t)]' < 0$ for $t \geq t_1$. If $a(t)[b(t)z'(t)]' < 0$, then there exists $t_2 \geq t_1$ such that

$$a(t)[b(t)z'(t)]' = -c, \quad c > 0 \quad \forall t \geq t_2.$$

$$\text{Thus } [b(t)z'(t)]' = \frac{-c}{a(t)}, \quad t \geq t_2.$$

Now by integrating from t_2 to t , we get

$$b(t) z'(t) = b(t_2) z'(t_2) - \int_{t_2}^t \frac{c}{a(s)} ds, \quad t \geq t_2,$$

Since $b(t) > 0$, then $z'(t) \rightarrow -\infty$ as $t \rightarrow \infty$, which may lead to $z(t)$ is eventually negative which contradicts the fact $z(t) > 0$. Hence we conclude that $a(t)[b(t)z'(t)]' > 0 \quad \forall t \geq t_2$. Now we claim that $z'(t) > 0$, since $a(t) > 0$ and $b(t) > 0$, then $[b(t)z'(t)]' > 0$, thus $z'(t)$ is monotone and of one signe (i.e. $z'(t) > 0$ or $z'(t) < 0$). Suppose that $z'(t) < 0$.

Now since $a(t)[b(t)z'(t)]' = c > 0$.

$$\text{Then } [b(t)z'(t)]' = \frac{c}{a(t)}, \quad t \geq t_2.$$

By integration, we get

$$b(t)z'(t) = b(t_2)z'(t_2) + \int_{t_2}^t \frac{c}{a(s)} ds, \quad t \geq t_2. \text{ Thus by } (H_1) \text{ we see that } b(t)z'(t) \rightarrow \infty \text{ as } t \rightarrow \infty,$$

which contradicts our assumption $z'(t) < 0$. Hence $z'(t) > 0$.

Theorem 2.1

Suppose that for each $l \geq t_0$ there exists a function $\phi(t, s, l) \in \Omega$ such that

$$\limsup_{t \rightarrow \infty} A \left[\frac{K}{b(s)} - a\phi^2; l, t \right] > 0 \tag{2.1}$$

where the operator $A[.; l, t]$ and the function $\phi = \phi(t, s, l)$ are defined by (1.5) and (1.6). Then Eq. (1.1) is oscillatory.

Proof

Suppose that there exists a nonoscillatory solution $x(t)$ such that $x(t) > 0 \forall t \geq t_1 \geq t_0$. Then $z(t) > 0$.

Now define
$$U(t) = \frac{a(t)[b(t)z'(t)]'}{b(t)z'(t)}$$

Then by Lemma 2.1 we see that $U(t)$ is well defined and $U(t) > 0$.

Differentiating $U(t)$, we obtain

$$U'(t) = \frac{(a(t)[b(t)z'(t)]')'}{b(t)z'(t)} - a(t) \left[\frac{[b(t)z'(t)]'}{b(t)z'(t)} \right]^2, \quad \forall t \geq t_1$$

Now, from (1.1) and (H_3) , we get

$$(a(t)[b(t)z'(t)]')' = -f(t, z(t), z'(t)) \leq -K z'(t) \tag{2.2}$$

Using (2.2), we have

$$U'(t) \leq -\frac{K}{b(t)} - \frac{U^2(t)}{a(t)}, \quad \forall t \geq t_1 \tag{2.3}$$

Applying the operator $A[.; l, t]$ to (2.3), we get

$$A[U'(s); l, t] \leq -A\left[\frac{K}{b(s)}; l, t\right] - A\left[\frac{U^2(s)}{a(s)}; l, t\right], \quad \forall t \geq t_1$$

Thus, by the properties of the operator $A[.; l, t]$, we obtain

$$\begin{aligned} A\left[\frac{K}{b(s)}; l, t\right] &\leq -A\left[\frac{U^2(s)}{a(s)} - 2U(s)\phi; l, t\right], \quad \forall t \geq t_1 \\ &= -A\left[\left(\sqrt{\frac{1}{a(s)}} U(s) - \sqrt{a(s)} \phi\right)^2; l, t\right] + A[a\phi^2; l, t] \leq A[a\phi^2; l, t] \end{aligned}$$

$$A \left[\frac{K}{b(s)} - a \varphi^2 ; l, t \right] \leq 0$$

$$\limsup_{t \rightarrow \infty} A \left[\frac{K}{b(s)} - a \varphi^2 ; l, t \right] \leq 0$$

This contradicts (2.1). Hence Eq. (1.1) is oscillatory.

Remark 2.1

If we choose

$$\Phi(t, s, l) = \sqrt{H_1(s, l)H_2(t, s)}, \quad H_1, H_2 \in X, \text{ then } \quad \varphi(t, s, l) = \frac{1}{2} \left[\frac{h_1^{(1)}(s, l)}{\sqrt{H_1(s, l)}} - \frac{h_2^{(2)}(t, s)}{\sqrt{H_2(t, s)}} \right].$$

Where $h_1^{(1)}(s, l) = \frac{1}{\sqrt{H_1(s, l)}} \frac{\partial H_1(s, l)}{\partial s}$, and $h_2^{(2)}(s, l) = \frac{-1}{\sqrt{H_2(t, s)}} \frac{\partial H_2(t, s)}{\partial s}$.

Then we have the following result

Corollary 2.1

Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ there exist two functions

$H_1, H_2 \in X$ such that

$$\limsup_{t \rightarrow \infty} \int_l^t H_1(s, l)H_2(t, s) \left[\frac{K}{b(s)} - \frac{a(s)}{4} \left(\frac{h_1^{(1)}(s, l)}{\sqrt{H_1(s, l)}} - \frac{h_2^{(2)}(t, s)}{\sqrt{H_2(t, s)}} \right)^2 \right] ds > 0$$

Theorem 2.2

Assume that there exists a function $g \in C^1([t_0, \infty), R)$ such that for some $\beta \geq 1, H, h \in X$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s)\Psi(s) - \frac{\beta}{4} a(s)v(s) h^2(t, s) \right) ds = \infty, \tag{2.4}$$

where $\Psi(t) = v(t) \left\{ \frac{K}{b(t)} + \frac{g^2(t)}{a(t)} - g'(t) \right\}$, $v(t) = \exp \left(-2 \int_{t_0}^t \frac{g(s)}{a(s)} ds \right)$.

Then Eq. (1.1) is oscillatory.

Proof

Let $x(t)$ be a nonoscillatory solution of (1.1), then there exists $t_1 \geq t_0$ such that

$x(t) > 0, x(\tau(t)) > 0, \quad \forall t \geq t_1$. As in the proof of Theorem 2.1 we have (2.2).

Now define

$$W(t) = v(t)\left[\frac{a(t)[b(t)z'(t)]'}{b(t)z'(t)} + g(t)\right] > 0, \quad t \geq t_1.$$

Then by Lemma 2.1 we see that $W(t)$ is well defined and $W(t) > 0$.

Differentiating $W(t)$, we obtain

$$W'(t) = v(t)\frac{(a(t)[b(t)z'(t)]')'}{b(t)z'(t)} - v(t)a(t)\left[\frac{[b(t)z'(t)]'}{b(t)z'(t)}\right]^2 + v'(t)\frac{a(t)[b(t)z'(t)]'}{b(t)z'(t)} + v'(t)g(t) + v(t)g'(t)$$

Using (2.2), we get

$$W'(t) \leq -\frac{K v(t)}{b(t)} - 2\frac{v(t)g(t)}{a(t)}\left(\frac{W(t)}{v(t)} - g(t)\right) - \frac{v(t)}{a(t)}\left(\frac{W(t)}{v(t)} - g(t)\right)^2 - 2\frac{v(t)g^2(t)}{a(t)} + v(t)g'(t)$$

$$W'(t) \leq -\Psi(t) - \frac{W^2(t)}{a(t)v(t)} \tag{2.5}$$

Multiplying (2.5) by $H(t, s)$ and integrating with respect to s from T to t , we get

$$\int_T^t H(t, s)\Psi(s)ds \leq H(t, T)W(T) - \int_T^t \left(\sqrt{\frac{H(t, s)}{\beta v(s) a(s)}} W(s) + \frac{1}{2}\sqrt{\beta v(s) a(s)} h(t, s) \right)^2 ds - \int_T^t \frac{(\beta - 1)H(t, s)}{\beta v(s) a(s)} W^2(s) ds + \int_T^t \frac{\beta}{4} v(s) a(s) h^2(t, s) ds$$

$$\int_T^t \left(H(t, s)\Psi(s) - \frac{\beta}{4} v(s) a(s) h^2(t, s) \right) ds \leq H(t, T)W(T)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s)\Psi(s) - \frac{\beta}{4} v(s) a(s) h^2(t, s) \right) ds \leq |W(t_3)| < \infty.$$

This contradicts (2.4). Hence Eq. (1.1) is oscillatory.

In Theorem 2.2, if we take $H(t, s) = (t - s)^{n-1}$ for $(t, s) \in D$, $n > 2$, then $h(t, s) = -(n - 1)(t - s)^{(n-3)/2}$. Hence we get the following result.

Corollary 2.1

Suppose that there exists a function $g \in C^1([t_0, \infty), R)$ such that for some integer $n > 2$ if

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t - s)^{n-3} \left((t - s)^2 \Psi(s) - \frac{\beta(n-1)^2}{4} v(s) a(s) \right) ds = \infty,$$

where $\Psi(t), v(t)$ are as in Theorem 2.2. Then every solution of Eq. (1.1) is oscillatory.

EXAMPLES

Example 3.1

Consider the D.E.

$$\left(\frac{1}{t^2}[t z'(t)]'\right)' + \left(\frac{1}{t} + \frac{2}{t^3}\right)z'(t) = 0 \tag{3.1}$$

Where $z(t) = x(t) + \frac{1}{2}x(t - \frac{1}{2})$

Here $a(t) = \frac{1}{t^2}$, $b(t) = t$, $f(t, z(t), z'(t)) = \left(\frac{1}{t} + \frac{2}{t^3}\right)z'(t)$, $p(t) = 1/2$, $\tau(t) = t - 1/2$

Choosing $\phi(t, s, l) = s - l$

It is clear that $\frac{f(t, z(t), z'(t))}{z'} = \frac{1}{t} + \frac{2}{t^3} \geq K > 0$ and $\phi(t, s, l) = \frac{1}{s - l}$.

Now

$$A \left[\frac{K}{b(s)} - a \phi^2 ; l, t \right] = \int_l^t (s - l)^2 \left[\frac{K}{s} - \frac{1}{s^2} \frac{1}{(s - l)^2} \right]$$

$$= \frac{t^2}{2} + -2lt + l^2 \ln t + \frac{1}{t} - \frac{l^2}{2} + 2l^2 - l^2 \ln l - \frac{1}{l}$$

$$\limsup_{t \rightarrow \infty} A \left[K - r \phi^2 ; l, t \right] = \infty > 0$$

Hence by Theorem 2.1 Eq. (3.1) is oscillatory.

One such solution of Eq. (3.1) is $x(t) = \sin t$.

REFERENCES

1. R. P. Agarwal, S. R. Grace and D. O'Regan, Oscillation Theory for Difference and functional Differential Equations, Kluwer, Dordrecht, 2000.
2. B. Baculiková, J. Džurina, Oscillation of third- order neutral differential equations” Mathematical and Computer Modelling 52(2010)215-226.
3. D. D. Bainov and D. P. Mishev, Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, New York, 1991.
4. J.Dzurina, Ethiraju Thandapani, Sivaraj Thandapani, “Oscillation of solutions to third-order half-linear differential equations.”Electronic journal of Differential Equations, Vol.2012(2012) 1-9.
5. L. H. Erbe, Q. K. Kong, and B. G. Zhang, “Oscillation Theory for Functional Differential Equations,” Marcel Dekker, New York, 1995.
6. Z. Han , Tongxing Li, Shurong Sun and Chenghui Zhang, ” Oscillation behavior of third-order neutral Emden-Fowler delay differential equations on timescales” Advances in Difference Equations Vol. 2010, Article ID 586312.
7. Y.V. Rogovchenko, F. Tuncay, Oscillation criteria for second order nonlinear differential equations with

- damping, *Nonlinear Anal.* 69 (2008) 208-221.
8. Tongxing Li, Chenghui Zhang, and Guojing Xing, "Oscillation of third-order neutral delay differential equations" *Abstract and Applied Analysis* Vol. 2012, Article Id69201.
 9. R. Xu, F. Meng, New Kameneve-type oscillation criteria for second order neutral nonlinear differential equations, *Appl. Math. Comput.* 188 (2007) 1364-1370.
 10. Z. Yu, Q. Wang, Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales, *J. Comput. Appl. Math.* 225 (2009) 531-540.
 11. H. Zhong-yue, Oscillations of a class of third order nonlinear neutral functional differential equations, *J. of East China Normal University (Natural Science)*, 1 (2012) Article ID1000-5641, 113-120.